Solutions to Extra Credit Problems

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1. Evaluate
$$\int_0^1 \left\{\frac{1}{x}\right\}^2 dx,$$
where $\{\alpha\} = \alpha - \lfloor\alpha\rfloor$ denotes the fractional part of α.

Answer: $\ln(2\pi) - \gamma - 1$, where γ is Euler's constant.

Proof. We remark that the integral exists because $0 \le \{1/x\}^2 \le 1$ for all $x \in (0, 1]$. Fix a positive integer N and consider
$$\int_{1/N}^1 \left\{\frac{1}{x}\right\}^2 dx.$$
Substituting $x \mapsto 1/x$ gives
$$\int_{1/N}^1 \left\{\frac{1}{x}\right\}^2 dx = \int_1^N \frac{\{x\}^2}{x^2}\, dx.$$
For each positive integer k, if $k \le x < k+1$ then $\{x\} = x - k$. Therefore,
$$\int_1^N \frac{\{x\}^2}{x^2}\, dx = \sum_{k=1}^{N-1} \int_k^{k+1} \frac{(x-k)^2}{x^2}\, dx = \sum_{k=1}^{N-1} \left(2 - \frac{1}{k+1} - 2k\ln(k+1) + 2k\ln k\right).$$
Note that
$$\sum_{k=1}^{N-1} 2 = 2N - 2 \qquad \text{and} \qquad \sum_{k=1}^{N-1} \frac{-1}{k+1} = -\ln N - \gamma_N + 1,$$
where $\gamma_N \to \gamma$ as $N \to \infty$. Furthermore,
$$\sum_{k=1}^{N-1} -2k\ln(k+1) + 2k\ln k = -2\sum_{k=1}^{N-1} [(k+1)\ln(k+1) - k\ln k] + 2\sum_{k=1}^{N-1} \ln(k+1) = -2N\ln N + 2\ln(N!).$$
From Stirling's approximation
$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N e^{\theta_N}$$
(where $\theta_N \to 0$ as $N \to \infty$), we have
$$2\ln(N!) = 2N\ln N - 2N + \ln(2\pi N) + 2\theta_N.$$
Altogether,
$$\sum_{k=1}^{N-1} \left(2 - \frac{1}{k+1} - 2k\ln(k+1) + 2k\ln k\right) = \ln(2\pi) - \gamma_N - 1 + 2\theta_N.$$
This gives
$$\int_{1/N}^1 \left\{\frac{1}{x}\right\}^2 dx = \ln(2\pi) - \gamma_N - 1 + 2\theta_N,$$
and taking $N \to \infty$ gives $\ln(2\pi) - \gamma - 1$. $\square$

2. Evaluate $\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k}$.

Answer: $1-\gamma$, where γ is Euler's constant.

Proof. Call the sum S. Insert the series definition for ζ and note that $\sum_{n=1}^{\infty} n^{-k} - 1 = \sum_{n=2}^{\infty} n^{-k}$ to get

$$S = \sum_{k=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{kn^k}.$$

These terms are all positive, so an application of Tonelli's theorem (equally applicable to both sums and integrals) gives

$$S = \sum_{n=2}^{\infty}\sum_{k=2}^{\infty} \frac{1}{kn^k}.$$

Recognize the series for the logarithm:

$$-\ln\left(1-\frac{1}{n}\right) = \sum_{k=1}^{\infty} \frac{1}{kn^k}.$$

Thus

$$S = \sum_{n=2}^{\infty}\left[-\ln\left(1-\frac{1}{n}\right)-\frac{1}{n}\right] = \sum_{n=2}^{\infty}\left[\ln n - \ln(n-1) - \frac{1}{n}\right]$$

Take a large positive integer N; we can telescope to get

$$\sum_{n=2}^{N}\left[\ln n - \ln(n-1) - \frac{1}{n}\right] = \ln N - H_N + 1,$$

where we use the notation $H_N = 1 + 1/2 + 1/3 + \cdots + 1/N$. It is well known that $H_N = \ln N + \gamma + O(1/N)$, so

$$S = \lim_{N\to\infty}(\ln N - H_N + 1) = \lim_{N\to\infty}(1-\gamma+O(1/N)) = 1-\gamma,$$

as claimed. □

3. Evaluate $\sum_{n=0}^{\infty} \binom{2n}{n}^{-1}$.

Answer: $4/3 + 2\pi\sqrt{3}/27$

Proof. There are many approaches—using arcsine, using hypergeometric functions, etc.—but we'll use the gamma and beta functions. Define the functions

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-}t\,dt \qquad \text{and} \qquad B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

It is easy to check (via integration by parts and mathematical induction) that $\Gamma(x+1) = x\Gamma(x)$ in general and that $\Gamma(n) = (n-1)!$ for integers n. Furthermore,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

which follows from simple manipulation of the integrals. Altogether, we have

$$\binom{2n}{n}^{-1} = \frac{(n!)^2}{(2n)!} = \frac{\Gamma(n+1)^2}{\Gamma(2n+1)} = \frac{(2n+1)\Gamma(n+1)^2}{\Gamma(2n+2)} = (2n+1)B(n+1,n+1).$$

Written in terms of the integrals,

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} = \sum_{n=0}^{\infty} (2n+1) \int_0^1 t^n (1-t)^n \, dt = \sum_{n=0}^{\infty} \int_0^1 (2n+1)(t-t^2)^n \, dt.$$

For $t \in [0,1]$ and $n \geq 0$, we have $(2n+1)(t-t^2)^n \geq 0$, so the monotone convergence theorem gives

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} = \int_0^1 \sum_{n=0}^{\infty} (2n+1)(t-t^2)^n \, dt.$$

The sum can be evaluated by differentiating the geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \Longrightarrow \qquad \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2},$$

so we have

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} = \int_0^1 \frac{1+t-t^2}{(1-t+t^2)^2} \, dt.$$

This integral can be evaluated to get the answer of $4/3 + 2\pi\sqrt{3}/27$. □

4. Let R be a nice region in $\mathbb{R}^2$ containing the origin and p be the Green function of R with singularity at zero. Assume that ∂R moves with normal velocity given by $V = \partial p / \partial n$ (with p changing so that it remains the Green function of R). Given a harmonic function h, prove that

$$\frac{d}{dt} \iint_R h \, dA = h(0).$$

Proof. The following is a kinematic relation for any such moving region; its proof can be found in most books on fluid mechanics.

$$\frac{d}{dt} \iint_R h \, dA = \oint_{\partial R} hV \, ds.$$

Since $V = \partial p / \partial n = \nabla p \cdot \mathbf{n}$, this becomes

$$\frac{d}{dt} \iint_R h \, dA = \oint_{\partial R} h \nabla p \cdot d\mathbf{n},$$

a 2D flux integral. To proceed, we use Green's first identity:

$$\iint_R (h\Delta p - p\Delta h) \, dA = \oint_{\partial R} (h\nabla p - p\nabla h) \cdot d\mathbf{n}.$$

To verify this, simply use the divergence theorem and product rule on the right-hand side of the equation. Since $\Delta h = 0$ and $p = 0$ on ∂R, Green's identity reduces to

$$\iint_R h\Delta p \, dA = \oint_{\partial R} h\nabla p \cdot d\mathbf{n} = \frac{d}{dt} \iint_R h \, dA.$$

Finally, by definition of the Green function, the area integral on the left is simply $h(0)$. □

5. Learn about Stokes' theorem for $\mathbb{C}$:

$$\iint_R \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \wedge dz = \oint_{\partial R} f \, dz$$

and use it to prove the Cauchy integral theorem. Discuss.

Proof. Literally an entire book can be written for this. We'll just stick to the basics; from Stokes' theorem for differential forms,

$$\oint_{\partial R} f\,dz = \iint_R d(f\,dz) = \iint_R \frac{\partial f}{\partial \bar{z}}\,d\bar{z} \wedge dz + \underbrace{\frac{\partial f}{\partial z}\,dz \wedge dz}_{=0}$$

The requirements on R and f are that f must be continuously differentiable and R must have a boundary with continuously differentiable parametrization. A basic result in complex analysis is that complex differentiable functions f must satisfy $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$. In terms of $\bar{z}$, this is equivalent to

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

So if f is complex differentiable (nontrivially, this is equivalent to 'analytic' when dealing with $\mathbb{C}$), then

$$0 = \iint_R \frac{\partial f}{\partial \bar{z}}\,d\bar{z} \wedge dz = \oint_{\partial R} f\,dz.$$

This is the Cauchy integral theorem. Cauchy's theorem can actually be proved without requiring that f be continuously differentiable, but that requires an entirely different approach. □